Disc 105

Wednesday, February 24, 2021 11:01 AM

3.2 Problems

Problem 1. *Use Neville's method to approximate $\sqrt{3}$ with the following functions and values:*

1. $f(x) = 3^x$ *and the values* $x_0 = -2, x_1 = -1, x_2 = 0, x_3 = 1, x_4 = 2$
2. $f(x) = \sqrt{x}$ *and the values* $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5$
3. *compare the accuracy of the approximations in parts (a) and (b)*

Problem 2. *Let $P_3(x)$ be the interpolating polynomial for the data $(0,0), (.5,y), (1,3), (2,2)$, find y if the coefficient of x^3 in $P_3(x)$ is 6.*

Theorem 3.5 Let f be defined at $x_0, x_1, \dots, x_k$ and let x_j and x_i be two distinct numbers in this set. Then

$$P(x) = \frac{(x - x_j)P_{0,1,\dots,j-1,j+1,\dots,k}(x) - (x - x_i)P_{0,1,\dots,i-1,i+1,\dots,k}(x)}{(x_i - x_j)}$$

is the kth Lagrange polynomial that interpolates f at the $k+1$ points $x_0, x_1, \dots, x_k$.

$\sqrt{3} = 3^{1/2} = f(1/2)$

We want to approximate $f(1/2)$ using polynomials of growing degree

i	x_i	$f(x_i)$
0	-2	$f(-2)$
1	-1	$f(-1)$
2	0	$f(0)$
3	1	$f(1)$
4	2	$f(2)$

$Q_{0,0}(1/2)$ 0th order poly using x_0

$Q_{1,0}(1/2)$ - - - - - x_1

$Q_{2,0}(1/2)$ - - x_2

$Q_{3,0}(1/2)$ x_3

$Q_{4,0}(1/2)$ x_4

$$\boxed{\begin{aligned} Q_{0,0} &= .\overline{1} \\ Q_{1,0} &= .\overline{3} \\ Q_{2,0} &= 1 \\ Q_{3,0} &= 3 \\ Q_{4,0} &= 9 \end{aligned}} \Rightarrow$$

Get 1st order polynomials.

$Q_{1,1}$: 1st order poly. using x_0, x_1

$Q_{2,1}$ x_1, x_2

$Q_{3,1}$ x_2, x_3

$Q_{4,1}$ x_3, x_4

$$Q_{1,1}(1/2) = \frac{(\frac{1}{2} - x_0)\,Q_{1,0}(1/2) - (\frac{1}{2} - x_1)\,Q_{0,0}(1/2)}{(x_1 - x_0)}$$

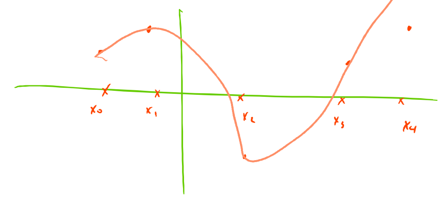
Get 2nd poly $Q_{2,2}$, $Q_{3,2}$, $Q_{4,2}$ 2nd order poly using x_0, x_1, x_2

$$Q_{2,2}(1/2) = \frac{(x - x_0)\,Q_{2,1}(1/2) - (x - x_2)\,Q_{1,1}(1/2)}{(x_2 - x_0)}$$

x_0, x_1, x_2

Neville's Method

Recursive algorithm to evaluate an interpolating poly @ a point.



Formula

$n-1$ degree poly $x_0, x_1, \dots, x_n$

if P_i interpolates all $\{x_k\}$ except x_i

& P_j interpolates all $\{x_k\}$ except x_j

if P_n interpolates on all

$$P_n(x) = \frac{(x - x_i)\,P_i - (x - x_j)\,P_j}{}$$

$x_j - x_i$

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1}), \quad (3.5)$$

$$a_k = f[x_0, x_1, x_2, \dots, x_k],$$

$$f[x_i] = f(x_i). \quad (3.7)$$

The remaining divided differences are defined recursively; the *first divided difference* of f with respect to x_i and x_{i+1} is denoted $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}. \quad (3.8)$$

$$P_n(x) = f(x_0) + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0)$$

3.3 Problems

Problem 3. *Use Newton forward-difference formula to construct interpolating polynomials of degree one, two, and three for the following data. Approximate the specfied value using each of the polynomials*

1. $f(.43)$ *if*

x_i	$f(x_i)$
0	1
.25	1.64872
.5	2.71828
.75	4.48169

Problem 4. *Show that the polynomial interpolating the following data has degree three.*

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

Theorem 3.3 Suppose $x_0, x_1, \dots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $\min\{x_0, x_1, \dots, x_n\}$, and the $\max\{x_0, x_1, \dots, x_n\}$ and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n), \quad (3.3)$$

er ways that the the Lagrange an be expressed, but st useful form and most closely agrees lard Taylor ror form.

where $P(x)$ is the interpolating polynomial given in Eq. (3.1).

Proof Note first that if $x = x_k$, for any $k = 0, 1, \dots, n$, then $f(x_k) = P(x_k)$, and choosing $\xi(x_k)$ arbitrarily in (a, b) yields Eq. (3.3).

case 1

x fixed

case 2

If $x \neq x_k$, for all $k = 0, 1, \dots, n$, define the function g for t in $[a, b]$ by

Key step 1

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)}$$

$$= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}.$$

Since $f \in C^{n+1}[a, b]$, and $P \in C^{\infty}[a, b]$, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$, we have

for all x_k's

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x_k - x_i)}{(x - x_i)} = 0 - [f(x) - P(x)] \cdot 0 = 0.$$

Moreover,

$$g(x) = f(x) - P(x) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x - x_i)}{(x - x_i)} = f(x) - P(x) - [f(x) - P(x)] = 0.$$

2nd key step

Thus, $g \in C^{n+1}[a, b]$, and g is zero at the $n+2$ distinct numbers $x, x_0, x_1, \dots, x_n$. By Generalized Rolle's Theorem 1.10, there exists a number ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$. So,

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)]\frac{d^{n+1}}{dt^{n+1}}\left[\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}\right]_{t=\xi}. \quad (3.4)$$

However, $P(x)$ is a polynomial of degree at most n, so the $(n+1)$st derivative, $P^{(n+1)}(x)$, is identically zero. Also, $\prod_{i=0}^{n}[(t - x_i)/(x - x_i)]$ is a polynomial of degree $(n+1)$, so

$$\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^{n}(x - x_i)}\right]t^{n+1} + (\text{lower-degree terms in } t),$$

and

$$\frac{d^{n+1}}{dt^{n+1}}\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} = \boxed{\frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)}}.$$

Equation (3.4) now becomes

$$0 = f^{(n+1)}(\xi) - 0 - [f(x) - P(x)]\frac{(n+1)!}{\prod_{i=0}^{n}(x - x_i)},$$

and, upon solving for $f(x)$, we have

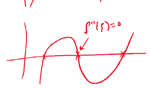
$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x - x_i). \qquad \blacksquare$$

$f(x) =$

$g(t, x) = f(t) - P(t) - (f(x) - P(x))\left(\prod \frac{(t - x_i)}{(x - x_i)}\right)$

$f'(\xi) = 0$

$f''(\xi) = 0$



$at^2 + bt + c$

$$\frac{(t - x_0)(t - x_1)(t - x_2)}{(x - x_0)(x - x_1)(x - x_2)} = \frac{1}{c}(t - x_0)(t - x_1)(t - x_2)$$

$(f(x) - P(x))\left(\quad \right) = f^{(n+1)}(\xi)$

$f(x) - P(x) = \left(\frac{\prod}{(n+1)!}\right)f^{(n+1)}(\xi)$

$\frac{d^2}{dx^2}\left(ax^2 + bx + c\right)$

$2ax$

$2a$